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Quantum deformation of the Poincaré supergroup and κ-deformed superspace

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Abstract. The classical r-matrix for the $N = 1$ super-Poincaré algebra, given by Lukierski *et al*, is used to describe the graded Poisson structure on the $N = 1$ Poincaré supergroup. The standard correspondence principle between the even (odd) Poisson brackets and (anti)commutators leads to the consistent quantum deformation of the super-Poincaré group with the deformation parameter q described by the fundamental mass parameter κ ($\kappa^{-1} = \ln q$). The κ-deformation of $N = 1$ superspace as dual to the κ-deformed supersymmetry algebra is discussed.

1. Introduction

Recently, in several papers [1–10], the quantum deformations of the $D = 4$ Poincaré algebra, which describes the relativistic symmetries, have been considered. Subsequently, we would like to stress here that during the last twenty years the supersymmetric extensions of relativistic symmetries has been one of the most studied ideas in the theory of fundamental interactions. We conclude, therefore, that it is natural to ask how to look for the quantum deformations of superalgebras or supergroups which describe the supersymmetric extensions of the four-dimensional spacetime symmetries.

The deformation of an $N = 1$ super Poincaré algebra with fourteen generators $I_A = (M_i, N_i, P_\mu, Q_\alpha, \overline{Q}_{\dot\alpha})$ ($A = 1, \ldots, 14$) can be studied in at least two different ways. The first considers the Hopf subalgebras of quantum superconformal algebra $U_q(SU(2, 2; 1))$. The complete description of this approach should take all possible quantum deformations of $SU(2, 2; 1)$§§. In the case studied so far (see [13]), the minimal Hopf subalgebra of $U_q(SU(2, 2; 1))$ containing deformed $N = 1$ super-Poincaré generators has 16 generators:

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§§ We would like to recall here that for the complexified conformal algebra, one can introduce the R-matrix with seven parameters [11]. The analogous general multiparameter deformations of quantum superalgebras were not studied in the literature (see, however, the partial results in [12]).

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14 generators of super-Poincaré algebra $\mathcal{P}_{4;1}$ as well as the dilatation generator D and the chiral generator A. We have, therefore,

$$U_q(SU(2,2;1)) \supset U_q(\mathcal{P}_{4;1} \ni\!\!\!\!+ (D \oplus A)) \tag{1.1}$$

i.e. in this way we obtain the quantum deformation of the $N = 1$ super-Weyl algebra.

The second considers the contraction of quantum super-de Sitter algebra $\mathcal{U}_q(OSp(1;4))$. It appears that such a method provides a genuine 14-generator quantum deformation of the $N = 1$ Poincaré superalgebra, the κ-deformed super-Poincaré algebra given first in [14] and described briefly in section 2.

In this paper, we shall study further the quantum deformation of the $N = 1$ super-Poincaré group given in [14]. From the κ-deformed super-Poincaré algebra, which is a non-commutative Hopf algebra, the non-trivial classical r-matrix can be extracted. Indeed, in [14] it has been shown that the graded-antisymmetric part of the first-order deformation-parameter coproducts $h \equiv 1/\kappa$ is given by

$$\delta(X) = \frac{1}{\kappa}[X \otimes \mathbf{1} + \mathbf{1} \otimes X, r] \tag{1.2}$$

$$r = N_i \wedge P_i - \frac{\mathrm{i}}{4} Q_\alpha \wedge \overline{Q}_{\dot\alpha} \equiv r^{AB} I_A \wedge I_B \tag{1.3}$$

where $A \wedge B \equiv A \otimes B - (-1)^{\eta(A)\eta(B)} B \otimes A$; $i = 1, 2, 3$; $\alpha = 1, 2$. The bitensor $r \in \hat{g} \otimes \hat{g}$ given by (1.3) describes the classical r-matrix for the $N = 1$ Poincaré superalgebra, where N_i denotes the boost generators, P_i denotes the three-momenta and Q_α, $Q_{\dot\alpha}$ describe the supercharges written as Weyl two-spinors. It appears that the classical r-matrix (1.3) satisfies the graded *modified* classical Yang–Baxter equation†, which permits us to introduce the non-trivial multiplication structure, determined by the cobracket (1.2), consistently onto the space g^* dual to g. Introducing the generators $Z_A \in \tilde{g}$ representing the supergroup parameters, one can define, on the functions $f(Z_A)$, the graded Poisson r-bracket

$$\{f, g\} = \{f, g\}_{\mathrm{R}} - \{f, g\}_{\mathrm{L}} \tag{1.4}$$

where $(a = R, L)$‡

$$\{f, g\}_a = (-1)^{\eta(A)\eta(B)} (\overleftarrow{\mathrm{D}}_A^{(a)} f) r^{AB} (\overrightarrow{\mathrm{D}}_B^{(a)} g) \tag{1.5}$$

† For the non-supersymmetric case see [15–17].

‡ In the supersymmetric case, one can introduce the left-hand and right-hand side derivatives

$$\overrightarrow{\mathrm{d}} f = \overrightarrow{\mathrm{d}} Z_A \frac{\overrightarrow{\partial} f}{\partial Z_A} \qquad \overleftarrow{\mathrm{d}} f = \frac{\overleftarrow{\partial} f}{\partial Z_A} \overleftarrow{\mathrm{d}} a \tag{A.1}$$

where $\overrightarrow{\mathrm{d}}^2 = \overleftarrow{\mathrm{d}}^2 = 0$, satisfying different Leibnitz rules

$$\overrightarrow{\mathrm{d}}(fg) = \overrightarrow{\mathrm{d}} f g + (-1)^{\eta(f)} f \overrightarrow{\mathrm{d}} g \qquad \overleftarrow{\mathrm{d}}(fg) = (-1)^{\eta(g)} \overleftarrow{\mathrm{d}} f g + f \overleftarrow{\mathrm{d}} g. \tag{A.2}$$

One gets that

$$\frac{\overrightarrow{\partial} f}{\partial Z_A} = (-1)^{\eta(f)\eta(Z_A)} \frac{\overleftarrow{\partial} f}{\partial Z_A}. \tag{A.3}$$

Using relations (A.3), one can write the Poisson r-bracket on a supergroup in four different ways, which differ by suitable *sign* factors. The choice (1.5) is the standard one.

where $\overset{\leftarrow(a)}{\mathrm{D}}_A$ denotes the left derivative which is right-invariant (left-invariant) under supergroup transformations for $a = R$ ($a = L$) and $\overset{\rightarrow(a)}{\mathrm{D}}_A$ denotes the right derivative which is right-invariant (left-invariant) for $a = R$ ($a = L$), respectively.

In section 3, we shall consider in greater detail the Poisson–Lie supergroup structure on the $N = 1$ Poincaré supergroup. It appears that for the choice of the r-matrix given by (1.3), the Poisson bracket (1.4) can be consistently quantized in a standard way, by the substitution of (graded) Poisson brackets by (anti-)commutators. In such a way, the supergroup parameters are promoted to the non-commuting generators of a quantum $N = 1$ Poincaré supergroup, with the coproduct rules described by the composition law of two $N = 1$ supersymmetry transformations.

It appears that after this quantization procedure, the Lorentz sector of the quantum $N = 1$ Poincaré supergroup is classical—in analogy with the case of the quantum Poincaré group, considered previously by Zakrzewski [18]. The deformation of the remaining generators of the quantum $N = 1$ Poincaré supergroup, describing translations and supertranslations, provides the κ-deformed $N = 1$ superspace, which is discussed in section 4. Finally, in section 5, we present an outlook and some unsolved problems.

2. $D = 4$ quantum super-Poincaré algebra

The κ-deformed $D = 4$ Poincaré superalgebra given in [14] has the structure of non-commutative and non-co-commutative Hopf superalgebra. It is described by the following set of relations.

(i) Lorentz sector ($M_{\mu\nu} = (M_i, N_i)$ where $M_i = \frac{1}{2}\epsilon_{ijk}M_{jk}$ describe the non-relativistic $O(3)$ rotations and N_i describe boosts).

Algebra.

$$[M_i, M_j] = i\epsilon_{ijk}M_k \qquad [M_i, N_j] = i\epsilon_{ijk}N_k \tag{2.1a}$$

$$[N_i, N_j] = -i\epsilon_{ijk}\left(M_k \cosh\frac{P_0}{\kappa} - \frac{1}{8\kappa}T_k \sinh\frac{P_0}{2\kappa} + \frac{1}{16\kappa^2}P_k(T_0 - 4M)\right) \tag{2.1b}$$

where ($\mu = 0, 1, 2, 3$)

$$T_\mu = Q^\alpha(\sigma_\mu)_{\alpha\dot\beta}\overline{Q}^{\dot\beta}. \tag{2.2}$$

Co-algebra.

$$\Delta(M_i) = M_i \otimes 1 + 1 \otimes M_i \tag{2.3a}$$

$$\begin{aligned}\Delta(N_i) = {} & N_i \otimes \mathrm{e}^{P_0/2\kappa} + \mathrm{e}^{-P_0/2\kappa} \otimes N_i + \frac{1}{2\kappa}\epsilon_{ijk}\left(P_j \otimes M_k\mathrm{e}^{P_0/2\kappa} + M_j\mathrm{e}^{-P_0/2\kappa} \otimes P_k\right)\\ & + \frac{\mathrm{i}}{8\kappa}(\sigma_i)_{\dot\alpha\beta}(\overline{Q}_{\dot\alpha}\mathrm{e}^{-P_0/4\kappa} \otimes Q_\beta\mathrm{e}^{P_0/4\kappa} + Q_\beta\mathrm{e}^{-P_0/4\kappa} \otimes \overline{Q}_{\dot\alpha}\mathrm{e}^{P_0/4\kappa}.\end{aligned} \tag{2.3b}$$

Antipodes.

$$\begin{aligned}S(M_i) &= -M_i\\ S(N_i) &= -N_i + \frac{3\mathrm{i}}{2\kappa}P_i - \frac{\mathrm{i}}{8\kappa}(Q\sigma_i\overline{Q} + \overline{Q}\sigma_i Q).\end{aligned} \tag{2.4}$$

(ii) Four-momenta sector $P_\mu = (P_i, P_0)$.

Algebra.

$$[M_i, P_j] = \mathrm{i}\epsilon_{ijk}P_k \qquad [M_j, P_0] = 0 \tag{2.5a}$$

$$[N_i, P_j] = \mathrm{i}\kappa\delta_{ij}\sinh\frac{P_0}{\kappa} \qquad [N_i, P_0] = \mathrm{i}P_i \tag{2.5b}$$

$$[P_\mu, P_\nu] = 0(\mu, \nu = 0, 1, 2, 3). \tag{2.5c}$$

Co-algebra.

$$\Delta(P_i) = P_i \otimes \mathrm{e}^{P_0/2\kappa} + \mathrm{e}^{-P_0/2\kappa} \otimes P_i \tag{2.6a}$$

$$\Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0. \tag{2.6b}$$

The antipode is given by the relation $S(P_\mu) = -P_\mu$.

(iii) Supercharges sector [14].

Algebra.

$$\begin{aligned} \{Q_\alpha, Q_{\dot\beta}\} &= 4\kappa\delta_{\alpha\beta}\sin\frac{P_0}{2\kappa} - 2P_i(\sigma_i)_{\alpha\dot\beta} \\ \{Q_\alpha, Q_\beta\} &= \{Q_{\dot\alpha}, Q_{\dot\beta}\} = 0 \end{aligned} \tag{2.7a}$$

$$[M_i, Q_\alpha] = -\tfrac{1}{2}(\sigma_i)_\alpha^{\ \beta}Q_\beta \qquad [M_i, Q_{\dot\alpha}] = -\tfrac{1}{2}(\sigma_i)_{\dot\alpha}^{\ \dot\beta}Q_{\dot\beta} \tag{2.7b}$$

$$[N_i, Q_\alpha] = -\frac{i}{2}\cosh\frac{P_0}{2\kappa}(\sigma_i)_\alpha^{\ \beta}Q_\beta \qquad [N_i, Q_{\dot\alpha}] = \frac{i}{2}\cosh\frac{P_0}{2\kappa}(\sigma_i)_{\dot\alpha}^{\ \dot\beta}Q_{\dot\beta} \tag{2.7c}$$

$$[P_\mu, Q_\alpha] = [P_\mu, Q_{\dot\beta}] = 0. \tag{2.7d}$$

Co-algebra.

$$\begin{aligned} \Delta(Q_\alpha) &= Q_\alpha \otimes \mathrm{e}^{P_0/4\kappa} + \mathrm{e}^{-P_0/4\kappa} \otimes Q_\alpha \\ \Delta(Q_{\dot\alpha}) &= Q_{\dot\alpha} \otimes \mathrm{e}^{P_0/4\kappa} + \mathrm{e}^{-P_0/4\kappa} \otimes Q_{\dot\alpha}. \end{aligned} \tag{2.8}$$

Antipodes.

$$S(Q_\alpha) = -Q_\alpha \qquad S(Q_{\dot\alpha}) = -Q_{\dot\alpha}. \tag{2.9}$$

On the basis of relations (2.3)–(2.7), one can single out the following features of the quantum superalgebra $\mathcal{U}_\kappa(\mathcal{P}_{4;1})$.

(i) The algebra coproducts and antipodes of Lorentz boosts N_i do depend on Q_α, $Q_{\dot\alpha}$, i.e. the κ-deformed Poincaré as well as the Lorentz sectors do not form Hopf subalgebras.

(ii) By putting $Q_\alpha = Q_{\dot\alpha} = 0$ into formulae (2.1)–(2.6), one obtains the κ-deformed Poincaré algebra considered in [4], i.e.

$$\mathcal{U}_\kappa(\mathcal{P}_{4;1})|_{Q_\alpha=Q_{\dot\alpha}=0} = \mathcal{U}_\kappa(\mathcal{P}_4).$$

(iii) From (2.5*c*), we see that the four-momenta commute. This property implies, by duality, the standard addition formula for the spacetime four-vectors (see section 4).

3. Poisson r-brackets for the $N = 1$ Poincaré supergroup and their quantization

The classical $N = 1$ Poincaré Lie superalgebra with the cobracket (1.2) describes the $N = 1$ Poincaré Lie super-bialgebra $(\hat{g}, \hat{\delta})$, which is called *coboundary* [17] due to relation (1.3) between the cobracket δ and the r-matrix.

The coboundary super-bialgebras with the r-matrix, satisfying the modified classical Yang–Baxter equation, describe infinitesimal Poisson–Lie supergroups, with the supergroup action $(Z_A, Z_B) \longrightarrow Z_A \circ Z_B$ consistent with the Poisson structure given by the r-Poisson bracket (1.5). These brackets satisfy the following properties.

(i) Graded antisymmetry.

$$\{f, g\} = -(-1)^{\eta(f)\eta(g)}\{g, f\}. \tag{3.1}$$

(ii) Graded Jacobi identity.

$$(-1)^{\eta(f)\eta(h)}\{f, \{g, h\}\} + (-1)^{\eta(g)\eta(h)}\{h, \{f, g\}\} + (-1)^{\eta(f)\eta(g)}\{g, \{h, f\}\} = 0. \tag{3.2}$$

(iii) Graded Leibnitz rules.

$$\begin{aligned} \{f, gh\} &= \{f, g\}h + (-1)^{\eta(f)\eta(g)}g\{f, h\} \\ \{fg, h\} &= f\{g, h\} + (-1)^{\eta(g)\eta(h)}\{f, h\}g. \end{aligned} \tag{3.3}$$

(iv) Lie–Poisson property. Let us write the coproduct induced by the composition law of two supergroup transformations as

$$\Delta(Z) = Z \overset{\circ}{\otimes} Z \tag{3.4}$$

where '$\overset{\circ}{\otimes}$' denotes that we take the composition rule described by '$\circ$' and replace the product by the tensor product. The Lie–Poisson property takes the form

$$\Delta\{f, g\} = \{\Delta(f), \Delta(g)\} \tag{3.5}$$

where the following rule for the multiplication of graded tensor products should be used:

$$(f_1 \otimes f_2)(g_1 \otimes g_2) = (-1)^{\eta(f_2)}(-1)^{\eta(g_1)} f_1 g_1 \otimes f_2 g_2. \tag{3.6}$$

In order to calculate explicitly the Poisson bracket (1.4), one can express the right- and left-invariant derivatives in terms of the ordinary derivatives, i.e. rewrite (1.4) as follows

$$\{f, g\} = f \frac{\overleftarrow{\partial}}{\partial Z_A} \omega^{AB}(z) \frac{\overrightarrow{\partial}}{\partial Z_B} g. \tag{3.7}$$

If we observe that

$$\overleftarrow{\mathrm{D}}{}^{(a)}_A = \frac{\overleftarrow{\partial}}{\partial Z^B} \overleftarrow{\mu}{}^{(a)B}_A(Z) \qquad \overrightarrow{\mathrm{D}}{}^{(a)}_A = \overrightarrow{\mu}{}^{(a)B}_A(Z) \frac{\overrightarrow{\partial}}{\partial Z^B} \tag{3.8}$$

where $\overleftarrow{\mu}^{(a)}$, $\overrightarrow{\mu}^{(a)}$ can be calculated by the differentiation of the composition formulae of the supergroup parameters Z_A, one obtains that ($L = +$, $R = -$)

$$\omega^{AB}(Z) = \overleftarrow{\mu}_C^{(+)A}(Z) r^{CD} \overrightarrow{\mu}_D^{(+)B}(Z) - \overleftarrow{\mu}_C^{(-)A}(Z) r^{CD} \overrightarrow{\mu}_D^{(+)B}(Z) \tag{3.9}$$

where the leading term at $Z = 0$ is linear and describes the cobracket of the $N = 1$ Poincaré bi-superalgebra $(\hat{g}, \hat{\delta})$ in accordance with relation (1.2).

The quantization of the $N = 1$ super-Poincaré algebra consists of two steps.

(i) Write (3.9) for the independent parameters Z^A (the generators of the algebra of functions on the supergroup $\mathcal{P}_{4;1}$)

$$\{Z^A, Z^B\} = \omega^{AB}(Z) \tag{3.10}$$

and calculate ω^{AB} by choosing the functions $\overleftarrow{\mu}^{(a)}$, $\overrightarrow{\mu}^{(a)}$ in (3.8), depending on the parametrization of the supergroup.

(ii) Quantize the Poisson bracket by the substitution

$$\{Z^A, Z^B\} \longrightarrow \begin{cases} \dfrac{1}{\mathrm{i}\hbar}[\hat{Z}^A, \hat{Z}^B]_- & \text{if } \eta(A)\cdot\eta(B) = 0 \\ \dfrac{1}{\mathrm{i}\hbar}[\hat{Z}^A, \hat{Z}^B]_+ & \text{if } \eta(A)\cdot\eta(B) = 1 \end{cases} \tag{3.11}$$

where $[\hat{A}, \hat{B}]_\pm = \hat{A}\hat{B} \pm \hat{B}\hat{A}$ and choose the ordering of the $\hat{Z}$-variables in ω^{AB} in such a way that the Jacobi identities are satisfied and the coproduct (3.4) is a homomorphism of the quantized superalgebra.

Let us recall the supergroup composition law (A is a 2×2 $Sl(2;\mathbb{C})$ matrix):

$$(X_\mu, \theta_\alpha, A_\alpha^\beta) \circ (X'_\mu, \theta'_\alpha, A'^\beta_\alpha) = \bigg(X_\mu + \Lambda_\mu^\nu(A) X'_\nu \\ + \frac{\mathrm{i}}{2}(\theta'^T A^{-1}\sigma^\mu\overline{\theta} - \theta^T\sigma^\mu (A^+)^{-1}\overline{\theta}'), \theta_\alpha + \theta'_\beta (A^{-1})_\alpha^\beta, A_\alpha^\gamma A'^\beta_\gamma \bigg). \tag{3.12}$$

Formulae (3.12) permits us to calculate the functions $\overleftarrow{\mu}^{(\pm)}$, $\overrightarrow{\mu}^{(\pm)}$ in formula (3.9). We obtain, for example, the following formulae for left-sided left-invariant super-derivatives:

$$\begin{aligned} \overrightarrow{\mathrm{D}}_\alpha^{(+)} &= (A^{-1})_\alpha^\beta \frac{\partial}{\partial\theta^\beta} + \frac{\mathrm{i}}{2}(A^{-1}\sigma^\mu\overline{\theta}_\alpha)\frac{\partial}{\partial X^\mu} \\ \overrightarrow{\mathrm{D}}_\alpha^{(+)\beta} &= A_\gamma^\beta \frac{\partial}{\partial A_\gamma^\alpha} \end{aligned} \tag{3.13}$$

and by conjugation

$$\begin{aligned} \overrightarrow{\mathrm{D}}_{\dot\alpha}^{(+)} &= (A^{-1})_{\dot\alpha}^{\dot\beta} \frac{\partial}{\partial\overline{\theta}^{\dot\beta}} + \frac{\mathrm{i}}{2}(\theta^T\sigma^\mu (A^+)^{-1})_{\dot\alpha}\frac{\partial}{\partial X^\mu} \\ \overrightarrow{\mathrm{D}}_{\dot\alpha}^{(+)\dot\beta} &= (A)_{\dot\gamma}^{\dot\beta} \frac{\partial}{\partial A_{\dot\gamma}^{\dot\alpha}}. \end{aligned} \tag{3.14}$$

Calculating the remaining invariant derivatives on the bosonic Poincaré subgroup and inserting the r-matrix (1.3) into formula (3.9), we obtain the following fundamental r-Poisson brackets for the coordinates $(X_\mu, A_\alpha^\beta, A_{\dot\alpha}^{\dot\beta}, \theta_\alpha, \theta_{\dot\alpha})$ on the $N = 1$ Poincaré supergroup†:

† We use the spinorial representation of the Lorentz generators, e.g. $L_i = \frac{1}{4}(\sigma_i)_\alpha^\beta L_\beta^\alpha + (\overline{\sigma}_i)_{\dot\alpha}^{\dot\beta} L_{\dot\beta}^{\dot\alpha}$.

(i) Lorentz sector (A_α^β, $A_{\dot\alpha}^{\dot\beta}$). The Lorentz subgroup parameters are classical, i.e.

$$\{A_\alpha^\beta, A_\gamma^\delta\} = \{A_\alpha^\beta, A_{\dot\gamma}^{\dot\delta}\} = \{A_{\dot\alpha}^{\dot\beta}, A_{\dot\gamma}^{\dot\beta}\} = 0. \tag{3.15}$$

(ii) Translations (X_μ). (We denote $\theta = \binom{\theta_1}{\theta_2}, \overline{\theta} = \binom{\dot\theta_1}{\dot\theta_2}$.)

$$\begin{aligned}\{X^i, X^j\} &= \frac{\mathrm{i}}{8\kappa}\theta^T\sigma^i(\mathbf{1}_2 - (AA^+)^{-1})\sigma^j\overline{\theta} - \frac{\mathrm{i}}{8\kappa}\theta^T\sigma^j(\mathbf{1} - (AA^+)^{-1})\sigma^i\overline{\theta} \\ \{X^0, X^j\} &= -\frac{\mathrm{i}}{\kappa}X^j + \frac{\mathrm{i}}{8\kappa}\theta^T[\sigma^j, (AA^+)^{-1}]\overline{\theta}\end{aligned} \tag{3.16}$$

$$\begin{aligned}\{A_\alpha^\beta, X^i\} &= \frac{1}{2\kappa}((A\sigma_n)_\alpha^\beta\Lambda_n^i(A) - (\sigma^i \cdot A)_\alpha^\beta) \\ \{A_\alpha^\beta, X^0\} &= \frac{1}{2\kappa}(A\sigma_i)_\alpha^\beta\Lambda_i^0(A).\end{aligned} \tag{3.17}$$

(iii) Supertranslations.

$$\{\theta^\alpha, \theta^\beta\} = \{\theta^{\dot\alpha}, \theta^{\dot\beta}\} = 0 \qquad \{\theta^\alpha, \theta^{\dot\beta}\} = \frac{\mathrm{i}}{2\kappa}(\mathbf{1} - AA^+)^{-1})^{\dot\beta\alpha} \tag{3.18}$$

$$\begin{aligned}\{X^i, \theta_\alpha\} &= \frac{1}{4\kappa}(\theta^T\sigma^i)_\gamma(\mathbf{1}_2 - (AA^+)^{-1})_\alpha^\gamma \\ \{X^0, \theta_\alpha\} &= -\frac{1}{4\kappa}\theta_\gamma^T(\mathbf{1}_2 + (AA^+)^{-1})_\alpha^\gamma\end{aligned} \tag{3.19}$$

$$\{A_\alpha^\beta, \theta^\gamma\} = \{A_{\dot\alpha}^{\dot\beta}, \theta^\gamma\} = 0. \tag{3.20}$$

In order to quantize the Poisson brackets (3.15)–(3.20), we perform the substitution (3.11). It appears that this substitution is consistent with Jacobi identities if we also keep the order of the coordinate generators on the right-hand side of (3.16) in the quantized case†. Furthermore, rewriting the composition law (3.12) as the coproduct rule for the coordinate generators gives

$$\begin{aligned}\Delta(X_\mu) &= X_\mu \otimes \mathbf{1} + \Lambda_\mu^\nu(A) \otimes X_\nu - \frac{\mathrm{i}}{2}(A_\alpha^{-1\beta}\sigma_{\beta\dot\gamma}^\mu\theta^{\dot\gamma} \otimes \theta^\alpha + \theta^\alpha\sigma_{\alpha\dot\beta}^\mu A_{\dot\gamma}^{-1\dot\beta} \otimes \theta^{\dot\gamma}) \\ \Delta(\theta_\alpha) &= \theta_\alpha \otimes \mathbf{1} + (A^{-1})_\alpha^\beta \otimes \theta_\beta \\ \Delta(A_\alpha^\beta) &= A_\alpha^\gamma \otimes A_\gamma^\beta.\end{aligned} \tag{3.21}$$

One can show that formulae (3.21) describe the homomorphism of the quantized superalgebra given in section 2. Adding the formulae for the antipodes

$$S(X^\mu) = -\Lambda_\nu^\mu(A^{-1})X^\nu \qquad S(A_\alpha^\beta) = (A^{-1})_\alpha^\beta \qquad S(\theta^\alpha) = -A_\beta^\gamma\theta^\beta \tag{3.22}$$

we see that we have obtained the complete set of relations describing the κ-deformation of the $N = 1$ Poincaré supergroup.

† For other relations (3.17)–(3.20), the problem does not occur due to the classical nature of the Lorentz sector (see (3.16)).

Let us observe that:

(i) if we put $A^{+}A = 1$, i.e. we consider the semidirect product $T_{4;4} \ni\!\!\!\!\!+ SU(2)$ of the quantum subgroup $T_{4,4}$ (quantum four-translations + quantum supertranslations) and $SU(2)$ describes the space rotations, the non-trivial κ-deformation occurs in only two relations (first, relation (3.17) and second, relation (3.19));

(ii) if we put $A = 1$, i.e. we consider the quantum subgroup $T_{4;4}$, we obtain the κ-deformed $N = 1$ superspace. It appears that only the commutator $[X^0, \theta_\alpha]$ is κ-deformed; and

(iii) putting $\theta^\alpha = \theta^{\dot{\alpha}} = 0$ into (3.15)–(3.17), one recovers the κ-deformed inhomogeneous $ISl(2;\mathbb{C})$ group, given in [19].

4. The κ-deformed $N = 1$ superspace

Let us recall first that for κ-deformed relativistic theory, with infinitesimal symmetries described by the κ-deformed Poincaré algebra [1, 2, 4, 6, 7], there are two different ways of introducing the Poincaré group and spacetime coordinates.

(i) Using formula (2.5c), one can consider the spacetime coordinates by considering *ordinary* Fourier transforms of the functions depending on the commuting four-momenta [4, 6, 20]. In such an approach, the spacetime coordinate operators $\hat{X}_\mu$ commute and are introduced as the operators satisfying the relations

$$[\hat{X}_\mu, \hat{P}_\nu] = \mathrm{i}\eta_{\mu\nu}. \tag{4.1}$$

(ii) Using the duality relation for Hopf algebras described by the scalar product on the quantum double with the following properties:

$$\begin{aligned} \langle \Delta(\hat{z}) | \hat{g}_1 \otimes \hat{g}_2 \rangle &= \langle \hat{z} | \hat{g}_1 \hat{g}_2 \rangle \\ \langle \hat{z}_1 \otimes \hat{z}_2 | \Delta(\hat{g}) \rangle &= \langle \hat{z}_1 \hat{z}_2 | \hat{g} \rangle \end{aligned} \tag{4.2}$$

we easily see that for the standard duality relation between $\hat{X}_\mu$ and $\hat{P}_\mu$ generators, non-co-commutative four-momenta (see (2.6)) imply the non-commutativity of the coordinates [18]

$$[\hat{X}^i, \hat{X}^j] = 0 \qquad [\hat{X}^0, \hat{X}^j] = \frac{1}{\kappa}\hat{X}^j \tag{4.3}$$

and commutativity of the four-momenta implies that

$$\Delta(\hat{X}^\mu) = \hat{X}^\mu \otimes 1 + 1 \otimes \hat{X}^\mu. \tag{4.4}$$

One can rewrite the coproduct formulae (2.6) and (4.4) as the addition formulae for the four-momenta

$$p_i^{(1+2)} = p_i^{(1)} \mathrm{e}^{p_0^{(2)}/2\kappa} + p_i^{(2)} \mathrm{e}^{-p_0^{(1)}/2\kappa} \qquad p_0^{(1+2)} = p_0^{(1)} + p_0^{(2)} \tag{4.5a}$$

and for the spacetime coordinates

$$x^\mu_{(1+2)} = x^\mu_{(1)} + x^\mu_{(2)}. \tag{4.5b}$$

If we introduce the following element of the quantum double describing the translation sector of κ-Poincaré ($\hat{X}^0 = -\hat{X}_0$, $\hat{X}^i = -\hat{X}_i$)†

$$G(\hat{X}^\mu; \hat{P}_\mu) = \mathrm{e}^{-\frac{1}{2}\hat{X}_0\otimes\hat{P}_0}\mathrm{e}^{\mathrm{i}\hat{X}_i\otimes\hat{P}_i}\mathrm{e}^{-\frac{1}{2}\hat{X}_0\otimes\hat{P}_0} \tag{4.6}$$

one can encode the additional formulae (4.5*a*)–(4.5*b*) into the following multiplication rules:

$$G(\hat{X}^\mu; p_\mu^{(1)})G(\hat{X}^\mu; p_\mu^{(2)}) = G(\hat{X}^\mu; p_\mu^{(1+2)}) \tag{4.7a}$$

$$G(x^\mu_{(1)}; \hat{P}_\mu)G(x^\mu_{(2)}; \hat{P}_\mu) = G(x^\mu_{(1+2)}; \hat{P}_\mu). \tag{4.7b}$$

We see, therefore, that relations (4.6) describe the generalization of Fourier-transform kernels to the case of the translation sector of the κ-Poincaré group, with the coproducts determining their multiplication rule.

Let us extend such a scheme to the $N = 1$ super-Poincaré case. The non-commutative Hopf algebra, describing the κ-deformed superspace, is obtained by the quantization of relations (3.16)–(3.19) with $A = 1$. One obtains

$$\begin{aligned}
&[\hat{X}^i, \hat{X}^j] = 0 \qquad [\hat{X}^0, \hat{X}^j] = \frac{1}{\kappa}\hat{X}^j \\
&\{\hat{\theta}^\alpha, \hat{\theta}^\beta\} = \{\hat{\theta}^\alpha, \hat{\theta}^{\dot\beta}\} = \{\hat{\theta}^{\dot\alpha}, \hat{\theta}^{\dot\beta}\} = 0 \\
&[\hat{X}^i, \hat{\theta}^\alpha] = [\hat{X}^i, \hat{\theta}^{\dot\alpha}] = 0 \\
&[\hat{X}^0, \hat{\theta}^\alpha] = -\frac{1}{2\kappa}\hat{\theta}^\alpha \qquad [\hat{X}^0, \hat{\theta}^{\dot\alpha}] = -\frac{1}{2\kappa}\hat{\theta}^{\dot\alpha}
\end{aligned} \tag{4.8}$$

and the coproducts (3.21) imply the following composition law in superspace:

$$\begin{aligned}
&\hat{\theta}^\alpha_{(1+2)} = \hat{\theta}^\alpha_{(1)} + \hat{\theta}^\alpha_{(2)} \qquad \hat{\theta}^{\dot\alpha}_{(1+2)} = \hat{\theta}^{\dot\alpha}_{(1)} + \hat{\theta}^{\dot\alpha}_{(2)} \\
&\hat{X}^\mu_{(1+2)} = X^\mu_{(1)} + X^\mu_{(2)} + \frac{\mathrm{i}}{2}(\sigma^\mu)_{\alpha\dot\beta}(\theta^{\dot\beta}_{(1)}\theta^\alpha_{(2)} - \theta^\alpha_{(1)}\theta^{\dot\beta}_{(2)}).
\end{aligned} \tag{4.9}$$

We recall that the κ-deformed $N = 1$ superalgebra is described by relations (2.7) and coproducts (2.8). The addition formula of the Grassmann-algebra-valued eigenvalues q_α, $q_{\dot\alpha}$ of the supercharges, induced by (2.8), is

$$\begin{aligned}
q_\alpha^{(1+2)} &= q_\alpha^{(1)}\mathrm{e}^{p_0^{(2)}/4\kappa} + q_\alpha^{(2)}\mathrm{e}^{-p_0^{(1)}/4\kappa} \\
q_{\dot\alpha}^{(1+2)} &= q_{\dot\alpha}^{(1)}\mathrm{e}^{p_0^{(2)}/4\kappa} + q_\alpha^{(2)}\mathrm{e}^{-p_0^{(1)}/4\kappa}.
\end{aligned} \tag{4.10}$$

If we introduce the following quantum counterpart of the finite supertranslation group elements in momentum as well as coordinate superspace:

$$G(p_\mu, q_\alpha, q_{\dot\alpha}) = \mathrm{e}^{-\frac{1}{2}\hat{X}_0 p_0}\mathrm{e}^{\mathrm{i}(\hat{X}^i p_i + \hat{\theta}^\alpha q_\alpha + \hat{\theta}^{\dot\alpha} q_{\dot\alpha})}\mathrm{e}^{-\frac{1}{2}\hat{X}_0 p_0} \tag{4.11a}$$

$$\tilde{G}(x_\mu, \theta_\alpha, \theta_{\dot\alpha}) = \mathrm{e}^{\mathrm{i}(x^\mu\hat{P}_\mu + \theta^\alpha Q_\alpha + \theta^{\dot\alpha} Q_{\dot\alpha})} \tag{4.11b}$$

† For the concepts of exponentiation of the generators of the quantum double consisting of quantum Lie algebra and a dual quantum Lie group see [22–24] where the exponentials (4.6) are called quantum T-matrices. The notion of the quantum T-matrix is related to the notion of the universal bicharacter of Woronowicz (see e.g. [21].

where $\tilde{P}_0 = 2\kappa \sinh \frac{P_0}{2\kappa}$ and $\tilde{P}_i = P_i$, we obtain the following multiplication laws:

$$G(p_\mu^{(1)}, q_\alpha^{(1)}, q_{\dot\alpha}^{(1)})G(p_\mu^{(2)}, q_\alpha^{(2)}, q_{\dot\alpha}^{(2)}) = G(p_\mu^{(1+2)}, q_\alpha^{(1+2)}, q_{\dot\alpha}^{(1+2)}) \tag{4.12a}$$

$$\tilde{G}(x_{(1)}^\mu, \theta_{(1)}^\alpha, \theta_{(1)}^{\dot\alpha})\tilde{G}(x_{(2)}^\mu, \theta_{(2)}^\alpha, \theta_{(2)}^{\dot\alpha}) = \tilde{G}(x_{(1+2)}^\mu, \theta_{(1+2)}^\alpha, \theta_{(1+2)}^{\dot\alpha}). \tag{4.12b}$$

Following the discussion for ordinary supersymmetry (see e.g. [25]), one can consider the objects (4.11*a*) and (4.11*b*) as describing the superfields in momentum superspace and in the usual (coordinate) superspace, respectively.

It should be mentioned that algebra (4.8) describes the superspace coordinates in the particular Lorentz frame ($\Lambda = 1$). If we allow non-trivial Lorentz transformations, the algebra of superspace coordinates is no longer closed and one should consider the full algebra given by (3.15)–(3.20).

5. Outlook

In this paper we presented a quantum κ-deformation of the $N = 1$ Poincaré supergroup, which is a non-commutative and non-co-commutative Hopf superalgebra. We would like to mention the following problems which deserve further study.

(i) It appears that for the non-semisimple Lie (super)algebras, the 'naive' quantization (see (3.11)) of the r-Poisson bracket may be very useful as a consistent quantization scheme. In [18], as well as in the case presented in this paper, the ambiguities related to the ordering of the right-hand side of the quantized r-Poisson brackets are resolved in a unique way. It would be interesting to classify the classical r-matrices for non-semisimple Lie (super)algebras and find out for which cases the 'naive' quantization of the r-Poisson bracket would lead to a consistent quantization†.

(ii) One can show that the κ-deformed $N = 1$ supersymmetry algebra ($Q_{\dot\alpha}$, Q_α, P_μ) as a Hopf superalgebra (see section 2) is dual to the Hopf superalgebra describing the $N = 1$ κ-deformed superspace (see section 4). It would be important to show that the whole $N = 1$ κ-deformed supergroup is dual (possibly modulo some nonlinear transformations of the generators) to the $N = 1$ κ-Poincaré superalgebra, given in [14]. We would like to stress that such duality for the κ-deformed Poincaré group given in [18] is not known.

(iii) It would be interesting to generalize the results of [14] and of this paper to $N > 1$. We would like to mention that complete N-extended Poincaré superalgebra, with $N(N-1)$ central charges, can be obtained by the construction of the superalgebra $OSp(2N; 4)$ [26]. Replacing the classical superalgebra $OSp(2N; 4)$ by its q-analogue $\mathcal{U}_q(OSp(2N; 4))$ and performing the quantum-de Sitter construction limit with the rescaling (2.3), one should obtain the quantum deformation of the N-extended super-Poincaré algebra. For obtaining the N-extended κ-deformed Poincaré supergroup, it is sufficient to extend the classical r-matrix (1.3) to $N > 1$ and follow the method presented in this paper.

(iv) Finally, an important question is the application of the κ-deformed Poincaré algebra as well as the superalgebra to physical models. It should be stressed that the κ-generalization of the free classical and quantum fields has already been proposed (for the κ-deformed Klein–Gordon field see [6] and for the κ-deformed Dirac field see [6, 27]). For the description of the κ-deformed gauge fields, it is important to describe the κ-deformed

† This programme is now under consideration; the classical r-matrices for simple quantum Lie-algebras and the 'naive' quantization of corresponding quadratic r-Poisson brackets are also being studied.

differential calculus, in particular the κ-deformed Cartan forms. For this purpose, the non-commutativity of the generators of the κ-Poincaré group as well as the generators of the κ-Poincaré supergroup should be put in R-matrix form. It is known, however, that for the quantum κ-Poincaré group, such R-matrices exist and *do not* satisfy the quantum Yang–Baxter equation. We see, therefore, that the formalism of differential calculus on the κ-Poincaré group and the κ-Poincaré supergroup goes beyond the standard formulations given, e.g., in [28, 29] and requires further consideration.

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